Licenciatura em Gestão

Operational Research Chapter 2

2018-2019

ISEG 100 1911-2011 ANOS

100 ANOS A PENSAR NO FUTURO

2. Simplex Method

2.1 Introduction

2.2 Augmented Form and Basic Feasible Solutions

2.3 Simplex Algorithm

Prototype Example 1

x_1 – no. batches of P1 produced per week (P1=8-foot glass door with aluminum framing)

x_2 – no. batches of P2 produced per week (P2=4×6 foot double-hung wood framed window)

Z – total profit per week (in thousands of dollars) from producing these two products

Linear Programming (LP) Model:

$$Max\ Z = 3x_1 + 5x_2$$

$$s.t. \begin{cases} x_1 & & \leq 4 \\ & 2x_2 & \leq 12 \\ 3x_1 + & 2x_2 & \leq 18 \\ & x_1, x_2 \geq 0 & \end{cases}$$

Linear Programming

Definitions I (Recall)

Solution of an LP – a vector of R^n which components are the values of the variables;

Feasible Solution (FS) – a solution that satisfies all the constraints (functional and sign);

Non Feasible Solution (NFS) – a solution that does not satisfy at least one of the constraints;

Feasible Region (FR) – the set of all feasible solutions;

Optimal Solution (OS) – a feasible solution that gives the best value to the objective function (OF) (the best value=maximum or minimum);

Optimal value – the value of the objective function at an optimal solution;

Binding constraint in a solution – a constraint that hold with equality at that solution;

To solve an LP is to determine the optimal solution (or solutions) and the optimal value or to conclude that an optimal solution does not exist and why.

Properties (Recall)

Prop 1: The Feasible Region of an LP problem is either an empty set or a convex set.

Prop 2: If the Feasible Region of an LP problem is nonempty and bounded then at least an optimal solution exists.

Prop 3: If an LP problem has optimum then at least one of its corner point feasible solutions (CPF) is an optimal solution.

Prop 4: Given an LP Problem with optimum, if a CPF has no adjacent CPF with a better value for the Objective Function then that point is an optimal solution.

LP – The Simplex Method

Prop 3: If an LP problem has optimum then at least one of its corner point feasible solutions (FCP) is an optimal solution. (Recall)

$$Max\ z = 3x_1 + 5x_2$$

$$\begin{cases} x_1 \le 4 & (3) \\ 2x_2 \le 12 & (4) \\ 3x_1 + 2x_2 \le 18 & (5) \\ x_1 \ge 0 & (1) \\ x_2 \ge 0 & (2) \end{cases}$$

5 constraints 2 binding

↓

maximum $C_2^5 = \frac{5!}{2!(5-2)!}$ solutions !

$$\begin{matrix} (3) \\ (4) \end{matrix} \begin{cases} x_1 = 4 \\ 2x_2 = 12 \end{cases} \begin{cases} x_1 = 4 \\ x_2 = 6 \end{cases}$$

Feasible ?

$$\begin{cases} 3x_1 + 2x_2 \le 18 & (5)\ \text{False} \\ x_1 \ge 0 & (1)\ \text{OK} \\ x_2 \ge 0 & (2)\ \text{OK} \end{cases}$$

NO!

Binding constraints

$x_1 \geq 0$	$x_2 \geq 0$	$x_1 \leq 4$	$2x_2 \leq 12$	$3x_1 + 2x_2 \leq 18$
Yes	Yes			
Yes		Yes		
Yes			Yes	
Yes				Yes
	Yes	Yes		
	Yes		Yes	
	Yes			Yes
		Yes	Yes	
		Yes		Yes
			Yes	Yes

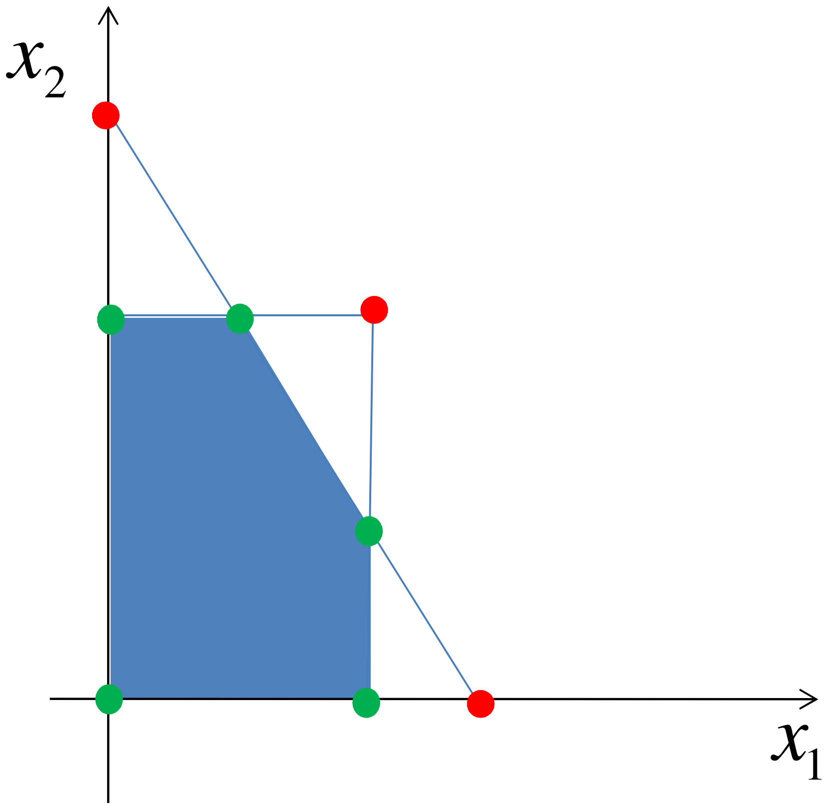
solution

Feasible / infeasible

x_1	x_2	
0	0	0
No sol.		
0	6	30
0	9	
4	0	12
No sol.		
6	0	0
4	6	
4	3	27
2	6	36

Evaluate the objective function value of the feasible solutions

Choose the best!



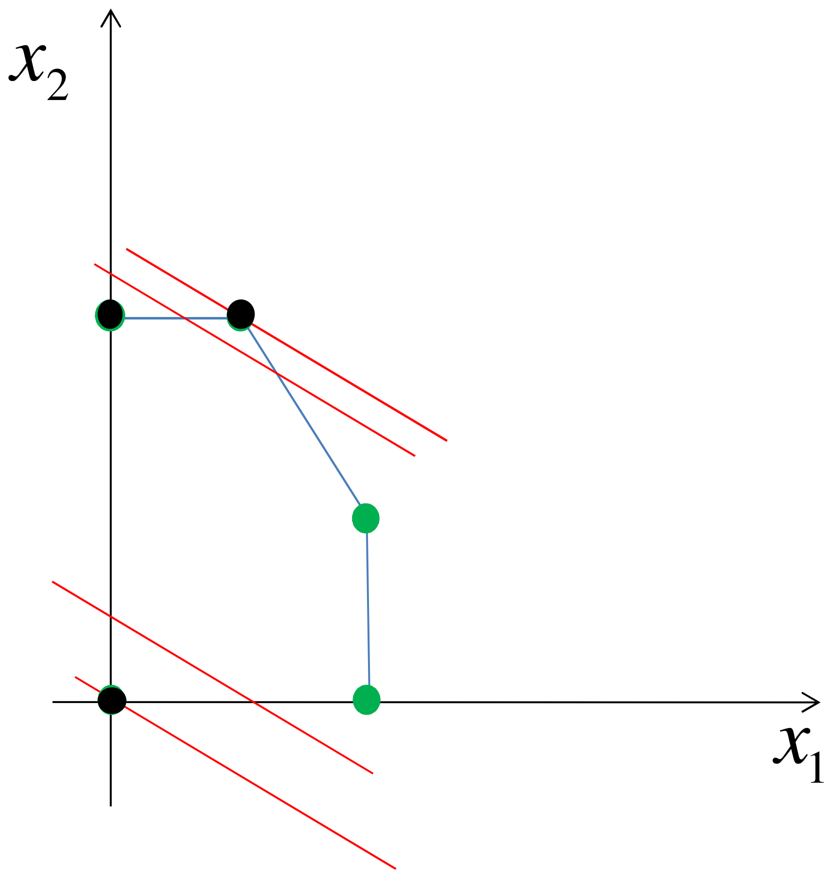
solution

Feasible / infeasible

x_1	x_2
0	0
No sol.	
0	6
0	9
4	0
No sol.	
6	0
4	6
4	3
2	6

LP – The Simplex Method

Prop 4: Given an LP Problem with optimum, if a FCP has no adjacent FCP with a better value for the OF then that point is an optimal solution.(Recall)



LP – The Simplex Method

In this course

Simplex method to solve LP problems in the standard form and nonnegative right-hand-sides .

Definitions

Standard form of an LP – a maximization problem
+ all functional constraints expressed by inequalities of the form ≤
+ all variables nonnegative (≥ 0);

Augmented form of an LP – LP model where
+ all variables nonnegative (≥ 0)
+ all functional constraints expressed by equations (using nonnegative variables - slack variables or surplus variables);

Augmented solution is a solution to the LP problem in the augmented form.

Properties

Prop 5: Every LP problem can be rewritten as an equivalent problem in the maximisation augmented form.

Writing an LP problem as a maximization augmented form problem

Objective: $\text{Min } Z = \sum_{j=1}^{n} c_j x_j \Leftrightarrow \text{Max}(-Z) = \sum_{j=1}^{n} (-c_j) x_j$ $\quad (\text{Min } Z = -\text{Max}(-Z))$

Constraints:

"≤": $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \wedge x_{n+i} \geq 0$

"≥ ": $\sum_{j=1}^{n} a_{ij} x_j \geq b_i \Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i \wedge x_{n+i} \geq 0$

Variables:

$x_j \leq 0 \Leftrightarrow x_j = -x'_j \quad (x'_j \geq 0)$

$x_j \text{ livre} \Leftrightarrow x_j = x'_j - x''_j \quad \left(\begin{cases} x'_j = \text{Max}\{0; x_j\} \geq 0 \\ x''_j = \text{Max}\{0; -x_j\} \geq 0 \end{cases} \right)$

Definitions

Consider a problem with m equations and ℓ (>m) nonnegative variables.

Set to zero $\ell - m$ nonbasic variables (NBV). If possible, in a unique way, solve the system of linear equations to obtain the value of the remaining m variables – basic variables (BV)

→ Basic Solution (BS).

Basic Feasible Solution (BFS) – is a BS with all variables verifying the signal constraints. Otherwise the solution is Basic InFeasible Solution (BIFS).

1 BFS → 1 corner point of the FR (Feasible Region)

1 corner point of the FR → at least one BFS

Adjacent basic solutions - only one different BV (In $\mathbb{R}^2$ - extreme points of a line segment)

LP – The Simplex Method

Prototype augmented form

$$Max\ z = 3x_1 + 5x_2$$
$$s.t.\begin{cases} x_1 & & \leq 4 \\ & 2x_2 & \leq 12 \\ 3x_1 & + 2x_2 & \leq 18 \\ x_1, & x_2 & \geq 0 \end{cases}$$

$$Max\ z = 3x_1 + 5x_2$$
$$s.t.\begin{cases} x_1 & & + x_3 & & & = 4 \\ & 2x_2 & & + x_4 & & = 12 \\ 3x_1 & + 2x_2 & & & + x_5 & = 18 \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0 \end{cases}$$

Binding constraints

x_1	x_2	x_3	x_4	x_5
0	0	4	12	18
0	-	0	-	-
0	6	4	0	6
0	9	4	-6	0
4	0	0	12	6
-	0	-	0	-
6	0	-2	12	0
4	6	0	0	-6
4	3	0	6	0
2	6	2	0	0

solution

Feasible / infeasible

x_1	x_2
0	0
No sol.	
0	6
0	9
4	0
No sol.	
6	0
4	6
4	3
2	6

LP –simplex algorithm

$$Max\ z = 3x_1 + 5x_2 = 0$$

$$s.t.\begin{cases} x_1 + x_3 = 4 & x_2 \leq \infty \\ 2x_2 + x_4 = 12 & x_2 \leq 6 \ (\frac{12}{2}) \\ 3x_1 + 2x_2 + x_5 = 18 & x_2 \leq 9 \ (\frac{18}{2}) \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

	Bas		coefficients							
iter	Var	Eq	Z	x_1	x_2	x_3	x_4	x_5	RHS	Op?
0	Z	0	1	-3	-5	0	0	0	0	NO
	x_3	1	0	1	0	1	0	0	4	
	x_4	2	0	0	2	0	1	0	12	
	x_5	3	0	3	2	0	0	1	18	

x_2 basic

$$Min\{\frac{12}{2}, \frac{18}{2}\}$$

x_4 non basic

LP –simplex algorithm

	Bas		coefficients							
iter	Var	Eq	Z	x_1	x_2	x_3	x_4	x_5	RHS	Op?
0	Z	0	1	-3	-5	0	0	0	0	NO
	x_3	1	0	1	0	1	0	0	4	
	x_4	2	0	0	2	0	1	0	12	
	x_5	3	0	3	2	0	0	1	18	
1	Z	0	1	-3	0	0	$5/2$	0	30	NO
	x_3	1	0	1	0	1	0	0	4	
	x_2	2	0	0	1	0	$1/2$	0	6	
	x_5	3	0	3	0	0	-1	1	6	
1	Z	0	1	0	0	0	$3/2$	1	36	YES
	x_3	1	0	0	0	1	$1/3$	$-1/3$	2	
	x_2	2	0	0	1	0	$1/2$	0	6	
	x_1	3	0	1	0	0	$-1/3$	$1/3$	2	

x_2 basic

$Min\{\frac{12}{2}, \frac{18}{2}\}$

x_4 non basic

x_1 basic

$Min\{\frac{4}{1}, \frac{6}{3}\}$

x_5 non basic

STOP $x_1^* = 2$

$x_2^* = 6$

$x_3^* = 2$

$x_4^* = x_5^* = 0$

$Z^* = 36$

LP –simplex algorithm (George Dantzig, 1947)

0. Consider an LP problem in the *standard* form with all RHS nonnegative.

Initialization

1. Write the problem in the *augmented* form
2. Determine an initial BFS: NBV = decision variables and solve the system;
   Write the simplex tableau; $k \longleftarrow 1$

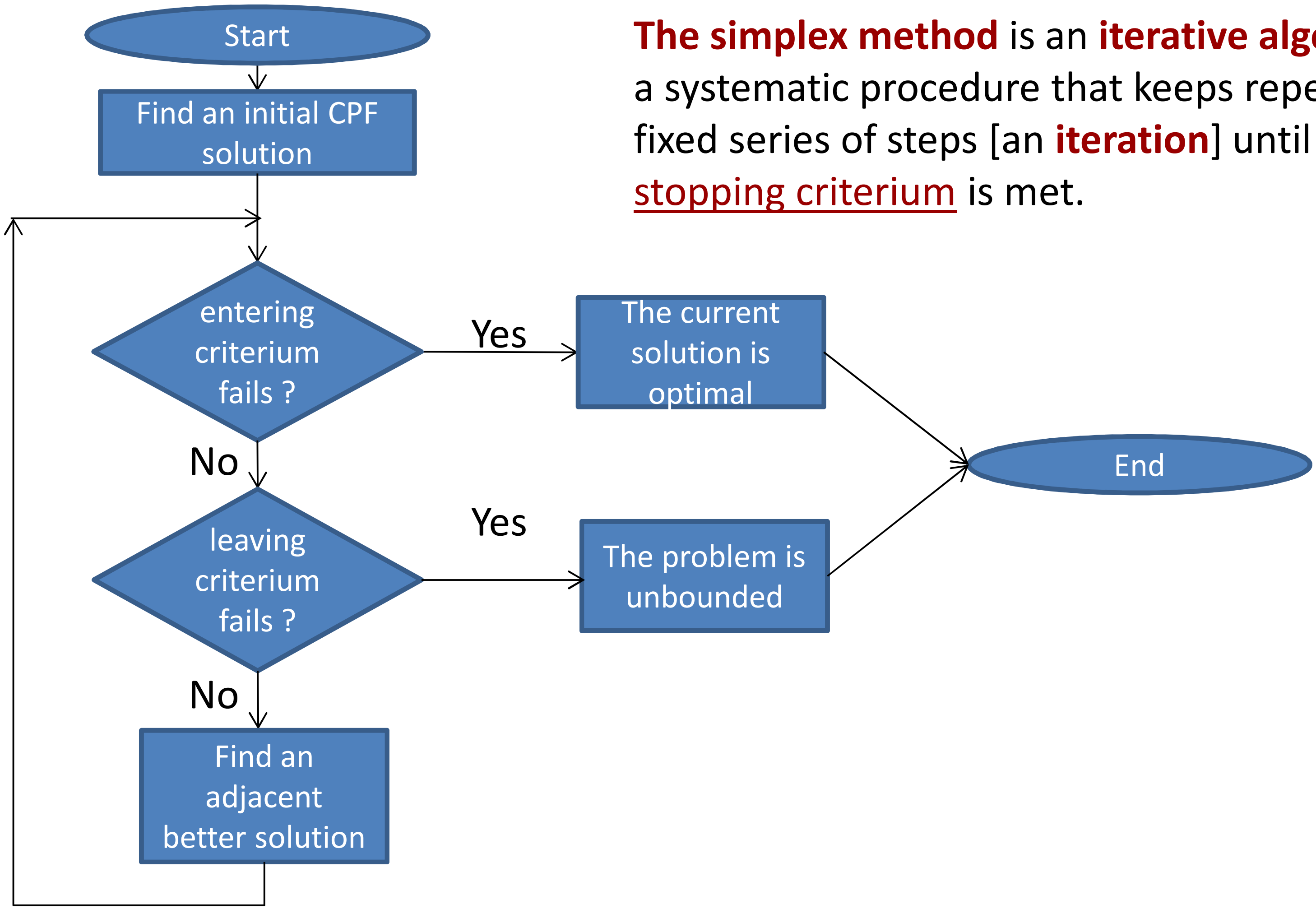
Iteration k

3. If the values in the Objective Function row ≥ 0, STOP (optimal basic solution found).
   Otherwise, EC: $\mathrm{Min}\{\text{coeffic.} < 0$ in the Objective Function row$\}$ [for the new BV x_p]
4. If all the coeffic. in the column of the $x_p$variable are ≤ 0, STOP (unbounded Obj. Function).
   Otherwise, LC: $\mathrm{Min}\left\{\frac{\text{RHS}}{\text{coeffic.}>0 \text{ in the column of } x_p}\right\}$ [for the new NBV x_r]
5. Update the simplex tableau ⇨ new BFS: x_p BV; x_r NBV;

   $k \longleftarrow k+1$; go to 3.

EC=entering criterium;LC=Leaving criterium.

LP – The Simplex Method



The simplex method is an **iterative algorithm** - a systematic procedure that keeps repeating a fixed series of steps [an **iteration**] until a stopping criterium is met.

3. Solve the following problems by the simplex algorithm.

a) $\text{Max } Z = 2x_1 + 3x_2 + 2x_3$

$$\text{s.t.:}\begin{cases} x_1 + 2x_2 + 3x_3 \leq 6 \\ x_1 + x_2 + x_3 \leq 10 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

b) $\text{Min } Z = -3x_1 + x_2 - 2x_3$

$$\text{s.t.:}\begin{cases} x_1 + x_2 \leq 3 \\ x_1 + 2x_2 + 2x_3 \leq 6 \\ 2x_1 + 2x_2 + x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

c) $\text{Max } Z = 3x_1 + 4x_2 + x_3$

$$\text{s.t.:}\begin{cases} x_1 + 2x_2 + x_3 \leq 5 \\ 2x_1 + 3x_2 + x_3 \leq 10 \\ 3x_1 + x_2 + x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

d) $\text{Max } Z = 4x_1 + 5x_2 + 3x_3$

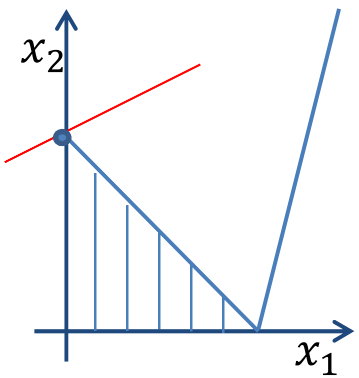
$$\text{s.t.:}\begin{cases} 7x_1 + 4x_2 + x_3 \leq 10 \\ 2x_1 + 3x_2 + 2x_3 \leq 4 \\ 3x_1 + 4x_2 + x_3 \leq 11 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Exercises (1.1 (book pg 32 1.3))

a)

$$Max\ z = x_1 + 2x_2$$

$$\text{s. t.} \begin{cases} x_1 - 2x_2 \le 3 \\ x_1 + x_2 \le 3 \\ x_1, x_2 \ge 0 \end{cases}$$



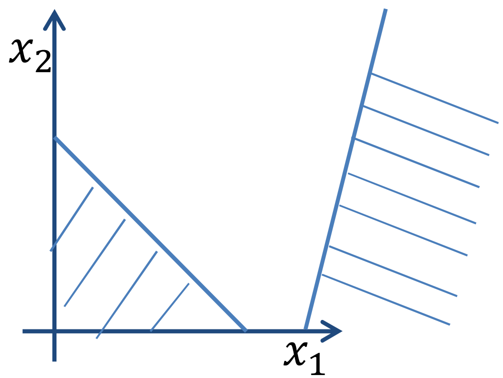
$z^* = 6$

x^*=(0,3)

b)

$$Max\ z = 3x_1 + 4x_2$$

$$\text{s. t.} \begin{cases} x_1 - 2x_2 \ge 4 \\ x_1 + x_2 \le 3 \\ x_1, x_2 \ge 0 \end{cases}$$

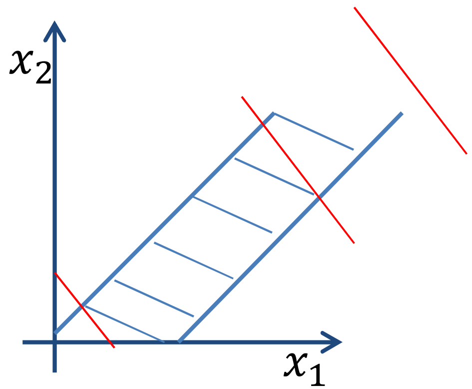


infeasible

c)

$$Max\ z = x_1 + x_2$$

$$\text{s. t.} \begin{cases} x_1 - x_2 \le 2 \\ x_1 - x_2 \ge 0 \\ x_1, x_2 \ge 0 \end{cases}$$



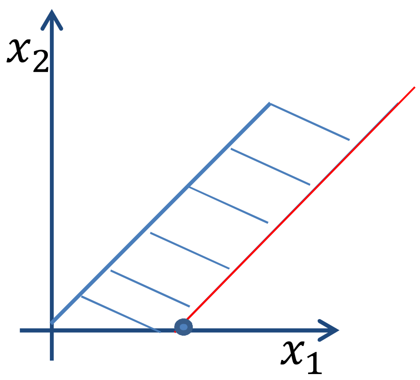
unbounded

Exercises (1.1 (book pg 32 1.3))

d)

$$Max\ z = x_1 - x_2$$

$$\text{s. t.} \begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$



$z^* = 2$

x^*=(2,0)

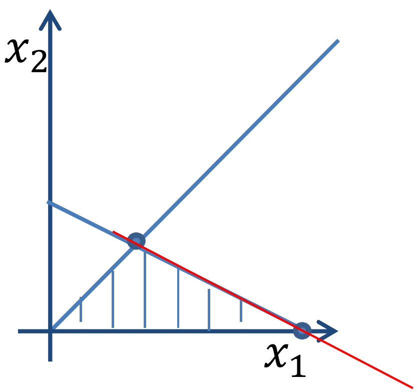
opt

Alternative optima

j)

$$Max\ z = 3x_1 + 6x_2$$

$$\text{s. t.} \begin{cases} x_1 + 2x_2 \leq 4 \\ x_1 - x_2 \geq 0 \\ x_1, x_2 \geq 0 \end{cases}$$



$z^* = 12$

x^*=(4,0)

opt

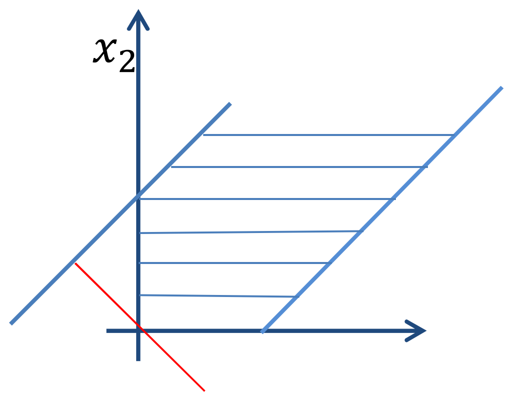
Alternative optima

x^*=($^4/_3$, $^4/_3$)

h)

$$min\ z = x_1 + x_2$$

$$\text{s. t.} \begin{cases} x_1 - x_2 \leq 2 \\ x_1 - x_2 \geq -2 \\ x_1, x_2 \geq 0 \end{cases}$$



$z^* = 0$

x^*=(0,0)

e) $Max\ z = -10x_1 - 5x_2$

$$\text{s. t.}\begin{cases} x_1 - x_2 \leq 5 \\ x_1 + \frac{8}{5}x_2 \geq -3 \\ x_1 \text{free} \\ x_2 \leq 0 \end{cases}$$

4.

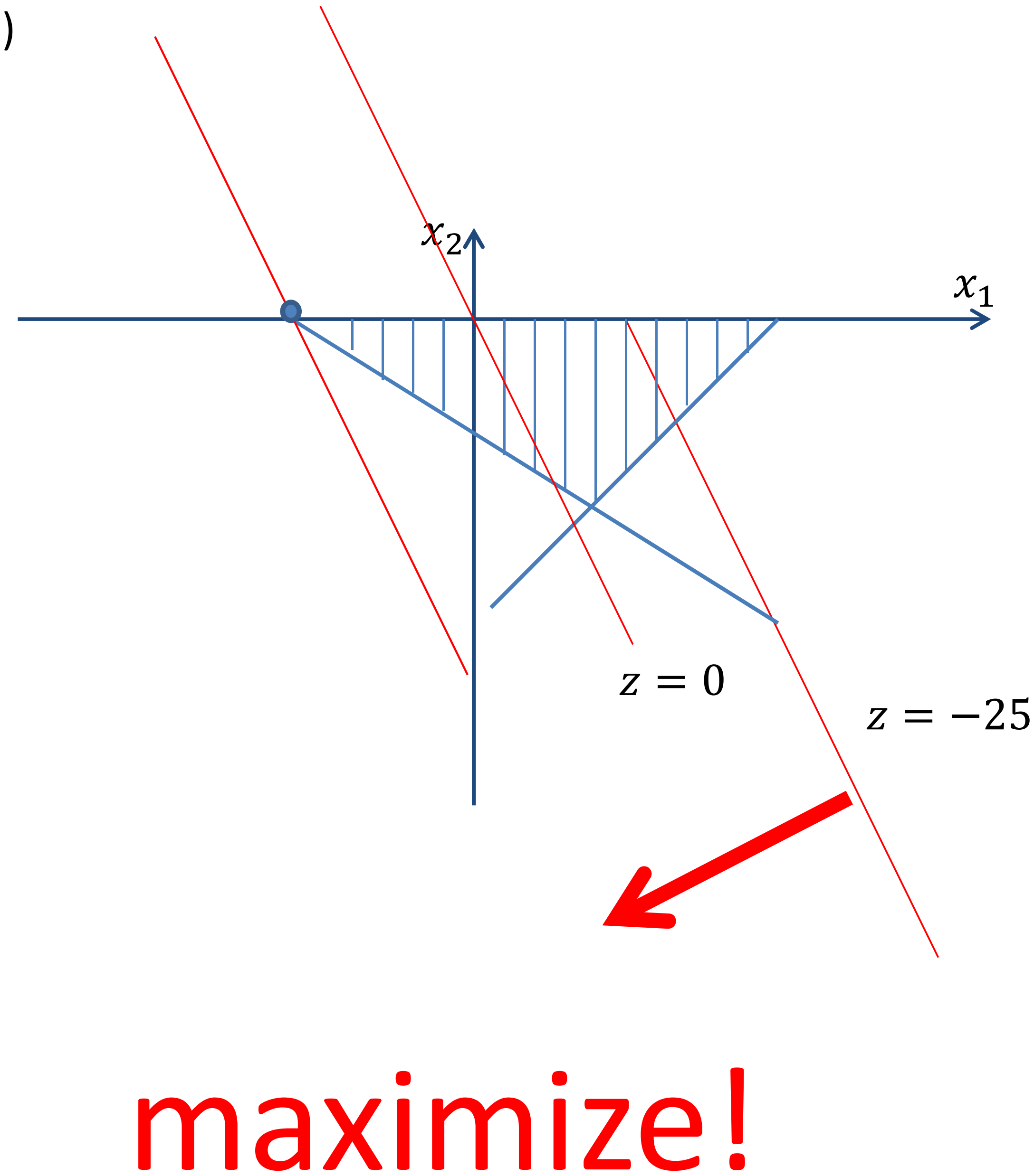
A firm has a plant with the capacity to work 70 hours a week and can produce three products (P1, P2 e P3). Each unit of P3 requires one hour of that capacity, while the unit production of P1 and P2 needs, respectively, the double and the triple of that time. The three products, when finished, are stored in a warehouse with $100\ m^3$ available. Each unit of product (P1, P2 or P3) requires$1\ m^3$. The gross unit margin achieved by each product is 10 (P1), 15 (P2) and 5 (P3).

a) Formulate an LP problem to maximize the total gross margin.
b) Find all the optimal solutions by simplex algorithm.

Exercises (1.1 (book pg 32 1.3))
e)

$$Max\ z = -10x_1 - 5x_2$$

$$\text{s. t.}\begin{cases} x_1 - x_2 \leq 5 \\ x_1 + \frac{8}{5}x_2 \geq -3 \\ x_1 \text{free} \\ x_2 \leq 0 \end{cases}$$



$z^* = 30$
x^*=(-3,0)

5. Consider the following LP problem P:

$$\text{Max } Z = x_1 - 3x_2$$

$$\text{s.t.:} \begin{cases} \frac{1}{3}x_1 + x_2 \leq 8 \\ x_1 - x_2 \leq 8 \\ x_1 \geq 0 \end{cases}$$

a) Solve P by the graphical method.

b) Determine the solution with the first constraint binding and $x_2 = 0$. Classify it.

c) Write P in the *standard* form and in the augmented form.

d) Solve P by the simplex algorithm.